REPRESENTATIONS OF REDUCTIVE GROUPS OVER FINITE RINGS AND EXTENDED DELIGNE-LUSZTIG VARIETIES

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Abstract. In a previous paper it was shown that a certain family of varieties suggested by Lusztig, is not enough to construct all irreducible complex representations of reductive groups over finite rings coming from the ring of integers in a local field, modulo a power of the maximal ideal. In this paper we define a generalisation of Lusztig's varieties, corresponding to an extension of the maximal unramified extension of the local field. We show in a particular case that all irreducible representations appear in the cohomology of some extended variety. We conclude with a discussion about reformulation of Lusztig's conjecture.

1. Introduction

Let F be a local field with finite residue field $\mathbb{F}_q$, ring of integers $\mathcal{O}_F$, and maximal ideal $\mathfrak{p}_F$. Let $\mathbf{G}$ be a connected reductive group over F, and set $G_r = \mathbf{G}(\mathcal{O}_F/\mathfrak{p}_F^r)$, for any integer $r \geq 1$.

Let F^{ur} be a maximal unramified extension of F, and let $\mathbf{G}_r = \mathbf{G}(\mathcal{O}_{F^{\mathrm{ur}}}/\mathfrak{p}_{F^{\mathrm{ur}}}^r)$. Then $\mathbf{G}_r$ carries a structure of affine algebraic group over the residue field $\mathbb{F}$ of F^{ur}, and is equipped with a morphism $\varphi : \mathbf{G}_r \to \mathbf{G}_r$ induced by the Frobenius automorphism in $\mathrm{Gal}(F^{\mathrm{ur}}/F)$, such that $\mathbf{G}_r^{\varphi} = G_r$.

In a previous version of [8], Lusztig conjectured (in the setting where F is of equal characteristic and $\mathbf{G}$ defined over $\mathbb{F}_q$) that every irreducible representation of G_r appears in the l-adic cohomology of a variety

$$X_x = \{g \in \mathbf{G}_r \mid g^{-1}\varphi(g) \in x\mathbf{U}_r\},$$

for some element $x \in \mathbf{G}_r$. The results in [11] show that this does not hold for the case $\mathbf{G} = \mathrm{SL}_2$, F of equal characteristic, $r = 2$, and q odd, but that the representations not accounted for by Lusztig's varieties, can in this case all be realised by a variety of a different kind. The latter variety was however constructed in a rather ad hoc manner, and it did not make it clear how to generalise it. It is therefore natural to seek a construction that extends Lusztig's conjecture in a more uniform and conceptual way. We believe that the ideas in this paper provide a first step in this direction.

The following is an outline of the contents. First we review Greenberg's theory of reduction of schemes over local rings, modulo some power of the maximal ideal. This provides the general theory underlying the construction and properties of the groups $\mathbf{G}_r$, and their generalisations.

Next, we define groups $\mathbf{G}_{L,r}$ for any finite Galois extension L/F^{ur}. The fact that G_r can be realised as the fixed points of $\mathbf{G}_{L,r}$ under the actions of the elements of the Galois group $\mathrm{Gal}(L/F)$, allows us to define a generalisation of the varieties of Lusztig, which we call extended Deligne-Lusztig varieties.

The main result of this paper is that every irreducible representation of dimension $(q^2-1)/2$ of the group G_2, where $\mathbf{G} = \mathrm{SL}_2$, F of equal characteristic, $r = 2$, and q odd, is realised by a certain extended Deligne-Lusztig variety, which is an analogue

of Lusztig's variety X_x for $x = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}$, where ε is a prime element in $\mathcal{O}_{F^{\mathrm{ur}}}$. We show that the ad hoc variety constructed in [11] is in fact isomorphic to the quotient of this extended variety, modulo a finite group. Together with the results of Lusztig (cf. [8], sect. 3) this shows that all irreducible representations of G_2 occur in the cohomology of some extended Deligne-Lusztig variety of a certain kind.

In the final section we speculate on how the construction of extended Deligne-Lusztig varieties may be used to reformulate Lusztig's conjecture. This perspective is still very rudimentary, which is shown by the large number of open questions.

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2. Reduction of schemes over local rings

In order to study the representation theory of reductive groups over the ring of integers in a local field modulo some power of the maximal ideal, it is very useful (perhaps essential) to view these groups as fixed point subgroups of certain algebraic groups which are reductions of the reductive group in question, modulo a power of the maximal ideal. Even though this theory of reduction can be described in elementary terms for affine varieties, it is more convenient to state it in the general case of schemes of finite type over certain local rings. The theory in this form, and all of the results in this section, are due to Greenberg (cf. [5], [6]).

Let R be a commutative noetherian local ring, with maximal ideal $\mathfrak{m}$ and residue field k, assumed to be perfect in the mixed characteristic case. Denote by X a scheme of finite type over R. Then there exists a functor $\mathcal{F}_R$ from the category of schemes of finite type over R, to schemes of over k, and a functor $\mathcal{G}_R$ from schemes over k to schemes over R, which is adjoint to $\mathcal{F}_R$ in the sense that

$$\mathrm{Hom}_R(\mathcal{G}_R(Y), X) \cong \mathrm{Hom}_k(Y, \mathcal{F}_R(X)),$$

for any scheme Y over k.

The functor $\mathcal{G}_R$ has the property that $\mathcal{G}_R(\mathrm{Spec}(k)) = \mathrm{Spec}(R)$, and so $X(R) \cong (\mathcal{F}_R X)(k)$, i.e. there exists a bijection between the points of X with values in R, and the points of $\mathcal{F}_R(X)$ with values in k.

An advantage of the functorial approach is that it is straightforward, once the categorical framework has been introduced, to show that the reduction of a group scheme is itself a group scheme. More precisely:

Proposition. *If X is a group scheme over R, then $\mathcal{F}_R X$ is a group scheme over k, and for every scheme Y over k, the bijection $(\mathcal{F}_R X)(Y) \cong (\mathcal{G}_R Y)(X)$ is an isomorphism of groups.*

Proof. This is Corollary 1, p. 639, and Corollary 4, p. 641 in [5]. ∎

Furthermore, the functor $\mathcal{F}_R$ preserves subschemes, takes schemes of finite type to schemes of finite type, separated schemes to separated schemes, and affine schemes to affine schemes, (cf. [5], Theorem, §4).

Now suppose we have a second commutative noetherian local ring R', and a homomorphism $\varphi : R \to R'$. Let X^φ be the scheme over R' obtained from X by extension of scalars. Then another consequence of the functorial construction is the existence of a connecting morphism

$$\mathcal{F}_\varphi(X) : \mathcal{F}_R X \longrightarrow \mathcal{F}_{R'} X^\varphi,$$

which is a group homomorphism if X is a group scheme, (cf. [5], §5).

Assume from now on that φ is surjective, and that the ideal $\operatorname{Ker}\varphi$ is annihilated by $\mathfrak{m}$. Let $\rho : R \to R/\mathfrak{m}$ be the canonical map. We call a scheme *simple* over R if

it has non-singular non-degenerate reduction mod $\mathfrak{m}$. Then we have the following results:

Proposition. *If X is simple over R, then $\mathcal{F}_\varphi(X)$ is surjective.*

Proof. Cf. Corollary 2, p. 262 in [6]. ∎

Proposition. *If X is simple over R and X^ρ is reduced and irreducible, then $\mathcal{F}_R X$ is reduced and irreducible.*

Proof. Cf. Corollary 2, p. 264 in [6]. ∎

We will now say something about projective limits. Suppose that R is complete with respect to its $\mathfrak{m}$-adic topology, i.e. $R \cong \varprojlim R/\mathfrak{m}^i$, with the connecting morphisms $\rho_{i,j} : R/\mathfrak{m}^i \to R/\mathfrak{m}^j$, for $i \geq j$. Let $X_{\mathfrak{m}^i}$ be the scheme over $R/\mathfrak{m}^i$ obtained from X by extension of scalars, and set $\mathcal{F}_{\mathfrak{m}^i} X := \mathcal{F}_{R/\mathfrak{m}^i} X_{\mathfrak{m}^i}$. Then it is shown in [5], §6 that the induced maps $\mathcal{F}_{\rho_{i,j}}(X)$ form a projective system, and there is a bijection

$$X(R) \longrightarrow \varprojlim (\mathcal{F}_{\mathfrak{m}^i} X)(k),$$

which is functorial in X. Moreover, if X is a group scheme, so is each $\mathcal{F}_{\mathfrak{m}^i} X$, each map $\mathcal{F}_{\rho_{i,j}}(X)$ is a homomorphism, and the above bijection is an isomorphism of groups.

In particular, the above results show that for R complete and X simple, the reduction mod $\mathfrak{m}$ maps the points $X(R)$ onto the points $(\mathcal{F}_{\mathfrak{m}} X)(k)$. This is a generalisation of Hensel's lemma.

The above results can also be used to prove a generalisation of Lang's theorem for p-adic group schemes. Let R be the ring of integers in a local field F with finite residue field, and maximal ideal $\mathfrak{p}$, and let $\widehat{R}^{\mathrm{ur}}$ be the ring of integers in $\widehat{F}^{\mathrm{ur}}$, the completion of a maximal unramified extension of F. Let $\varphi \in \mathrm{Gal}(\widehat{F}^{\mathrm{ur}}/F)$ be the Frobenius element. Then φ restricts to an automorphism of $\widehat{R}^{\mathrm{ur}}$ over R. For any group scheme G over R, we also denote by φ the induced map on $G(\widehat{R}^{\mathrm{ur}})$.

Proposition. *Let G be a group scheme simple over R, whose reduction mod $\mathfrak{p}$ is connected. Then the mapping $g \mapsto g^{-1}\varphi(g)$ of $G(\widehat{R}^{\mathrm{ur}})$ into itself, is surjective.*

Proof. Cf. [6], §3. ∎

3. Extended Deligne-Lusztig varieties

For any discrete valuation field F we denote by $\mathcal{O}_F$ its ring of integers, and by $\mathfrak{p}_F$ its maximal ideal. For any integer $r \geq 1$ we use the notation $\mathcal{O}_{F,r}$ for the quotient ring $\mathcal{O}_F/\mathfrak{p}_F^r$.

Let F be a local field with finite residue field $\mathbb{F}_q$. We fix an algebraic closure of F in which all algebraic extensions are taken. Denote by F^{ur} the maximal unramified extension of F with residue field $\mathbb{F}$, an algebraic closure of $\mathbb{F}_q$.

Let L_0 be a finite totally ramified Galois extension of F, and set $L = L_0^{\mathrm{ur}}$, with $\Gamma = \mathrm{Gal}(L/F)$. Then L is a henselian discrete valuation field with algebraically closed residue field $\mathbb{F}$. We have the relation $\mathfrak{p}_F\mathcal{O}_L = \mathfrak{p}_L^e$, where e is the ramification index, and since L_0/F is totally ramified we have $e = [L_0 : F]$. We may identify L with a finite extension of F^{ur} (cf. [4] ch. II, sect. 4), and thus the residue field of L is the same as that of F^{ur}.

Assume that X is a scheme of finite type over $\mathcal{O}_L$. In the context of the preceding section, we take $\mathcal{O}_L$ for the ring R. For any integer $r \geq 1$, we define

$$X_{L,r} = \mathcal{F}_{\mathcal{O}_{L,r}}(X \underset{\mathcal{O}_L}{\times} \mathcal{O}_{L,r}).$$

Conforming to the notation of [8] and [11], we set $X_r = X_{F^{\mathrm{ur}},r}$. Since L and F^{ur} have the same residue field, we have $X_{L,1} = X_1$.

Let $\mathbf{G}$ be a connected reductive affine algebraic group over F. Then $\mathbf{G}$ can be identified with its corresponding affine group scheme of finite type over F. Extending scalars to L, and using the inclusion $\mathcal{O}_L \to L$, we consider $\mathbf{G}$ as a group scheme over $\mathcal{O}_L$.

We shall assume that $\mathbf{G}_1$, the reduction of $\mathbf{G}$ modulo $\mathfrak{p}_{F^{\mathrm{ur}}}$, is a connected reductive group over the residue field $\mathbb{F}$. This condition means that for all $r > 1$, we have compatibility with the case $r = 1$. The condition is satisfied in particular when $\mathbf{G}$ is a Chevalley group.

The condition implies that $\mathbf{G}$ is simple over $\mathcal{O}_L$, so according to the results of the previous section, each $\mathbf{G}_{L,r}$ is an irreducible reduced affine group scheme of finite type over $\mathbb{F}$, i.e. a connected affine algebraic group.

Every automorphism $\sigma \in \Gamma$ stabilises $\mathcal{O}_L$ and $\mathfrak{p}_L^r$, respectively (cf. [4], chap. II, Lemma 4.1). Therefore, each $\sigma \in \Gamma$ defines a morphism of $\mathcal{O}_F$-algebras $\sigma : \mathcal{O}_{L,r} \to \mathcal{O}_{L,r}$. Thus, by the results of the previous section, $\mathbf{G}_{L,r}$ carries a natural structure of linear algebraic group over the residue field $\mathbb{F}$, and each $\sigma \in \Gamma$ induces a homomorphism $\sigma : \mathbf{G}_{L,r} \to \mathbf{G}_{L,r}$ with respect to this structure. In the following, we will use φ to denote both the Frobenius element in $\mathrm{Gal}(L/L_0)$, and its lift to Γ. Note that in compliance with this notation, Frobenius morphisms on algebraic groups will in this paper always be denoted by φ.

Let G_r denote the finite group of $\mathbb{F}_q$-points of the variety $\mathcal{F}_{\mathcal{O}_{F,r}}(\mathbf{G} \times_{\mathcal{O}_F} \mathcal{O}_{F,r})$. In [8] and [11], the group G_r was identified with the fixed points of $\mathbf{G}_r$ under the Frobenius map. However, this is not the only way to realise G_r as a group of fixed points of an algebraic group. The following assertion makes this more precise.

Lemma 1. *For every $r \geq 1$, we have $G_r = \mathbf{G}_{L,r'}^{\Gamma}$ if and only if $(r-1)e < r' \leq re$.*

Proof. For any $r' \geq 1$ it is clear that $\mathcal{O}_{L,r'}^{\Gamma} = \mathcal{O}_{F,r}$, where r is the largest integer such that $\mathcal{O}_{F,r} \subseteq \mathcal{O}_{L,r'}$. This implies that $\mathbf{G}_{L,r'}^{\Gamma} = G_r$, where r is the largest integer such that $G_r \subseteq \mathbf{G}_{L,r'}$. Now this happens exactly when $\mathfrak{p}_F^{r-1}\mathcal{O}_L \supseteq \mathfrak{p}_L^{r'-1} \not\subseteq \mathfrak{p}_F^{r}\mathcal{O}_L$, i.e. when $(r-1)e \leq r'-1 < re$, or equivalently $(r-1)e < r' \leq re$. ∎

Let Δ be a subset of Γ, and denote by $\{Y_\sigma \mid \sigma \in \Delta\}$ a family of locally closed subsets of $\mathbf{G}_{L,r}$. For each such family, we can define a variety

$$X = \{g \in \mathbf{G}_{L,r} \mid g^{-1}\sigma(g) \in Y_\sigma,\ \forall\, \sigma \in \Delta\}.$$

Let $\langle\Delta\rangle$ denote the subgroup of Γ, generated by Δ. Then the group $\mathbf{G}_{L,r}^{\Delta} = \mathbf{G}_{L,r}^{\langle\Delta\rangle}$ clearly acts on X by left multiplication. If each Y_σ is normalised by a subgroup T_σ of $\mathbf{G}_{L,r}$, then there is an action of $\bigcap_{\sigma\in\Delta}(T_\sigma)^\sigma$ on the variety, by right multiplication. If each Y_σ is stable under both left and right multiplication by a subgroup U_σ of $\mathbf{G}_{L,r}$, and each U_σ is σ-stable, then clearly $\bigcap_{\sigma\in\Delta} U_\sigma$ acts on X by right multiplication. Moreover, if each Y_σ is $\langle\Delta\rangle$-stable, then we have an action of $\langle\Delta\rangle$ on the variety.

Note that the varieties $\tilde{X}(\dot{w})$ of Deligne and Lusztig (cf. [2]) appear as special cases of the above construction. Namely, they are given by the specifications $r = 1$, $L = F^{\mathrm{ur}}$, $\Delta = \{\varphi\}$, and $Y_\varphi = \dot{w}\mathbf{U}$, where $\mathbf{U}$ is the unipotent radical of a Borel subgroup, and $\dot{w}$ is a lift of an element w in the Weyl group. Under similar assumptions, but with $r \geq 1$ and $Y_\varphi = \dot{w}\mathbf{U}_r$, we obtain the varieties considered by Lusztig in [7], §4 and [8].

Remark. We call the varieties defined in this section "extended Deligne-Lusztig varieties", both because they correspond to an extension of the maximal unramified extension, and because there are at least three other generalisations of (certain) Deligne-Lusztig varieties, neither of which is in the direction given here. One of

these is the varieties of Deligne associated to elements in certain braid monoids (cf. [1]); another is the affine Deligne-Lusztig varieties of Kottwitz and Rapoport (cf. [9]), and a third is the varieties of Digne and Michel [3], defined with respect to not necessarily connected, reductive groups.

4. An example

Let $\mathbf{G} = \mathrm{SL}_2$ and suppose that F is of positive characteristic with q odd. Let ε denote a prime element in F, and take $L_0 = F[\sqrt{\varepsilon}]$. Then Γ is topologically generated by two elements: the Frobenius automorphism φ, and an involution σ, given by $\sigma(a_0 + a_1\sqrt{\varepsilon}) = a_0 - a_1\sqrt{\varepsilon}$.

By Lemma 1, the smallest value of r for which $\mathbf{G}_{L,r}^{\Gamma} = G_2$, is $r = 3$. Thus from now on, we assume that $r = 3$. We let $\Delta = \{\varphi, \sigma\}$, and specify the family $\{Y_\varphi, Y_\sigma\}$ so that

$$Y_\varphi = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix} \mathbf{U}_{L,3}, \qquad Y_\sigma = \begin{pmatrix} 1 & 0 \\ \sqrt{\varepsilon} & 1 \end{pmatrix} \mathbf{U}_{L,3}.$$

Note that since $\mathbf{U}_{L,3}$ is closed, the same holds for the translation by any element in $\mathbf{G}_{L,3}$. With the above specifications, the resulting extended Deligne-Lusztig variety is

$$X_L := \left\{ g \in \mathbf{G}_{L,3} \mid g^{-1}\varphi(g) \in \left(\begin{smallmatrix} 1 & 0 \\ \varepsilon & 1 \end{smallmatrix}\right)\mathbf{U}_{L,3}, \ \ g^{-1}\sigma(g) \in \left(\begin{smallmatrix} 1 & 0 \\ \sqrt{\varepsilon} & 1 \end{smallmatrix}\right)\mathbf{U}_{L,3} \right\}.$$

Note that this variety carries a left action of the group G_2, and right actions of the groups

$$\mathbf{U}_2^1 = \left\{ \begin{pmatrix} 1 & x\varepsilon \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{F} \right\}, \quad A = \left\{ \begin{pmatrix} \pm 1 + a\varepsilon & 0 \\ 0 & \pm 1 - a\varepsilon \end{pmatrix} \mid a \in \mathbb{F}_q \right\}.$$

We wish to describe the variety X more explicitly. If we let

$$g = \begin{pmatrix} a_0 + a_1\sqrt{\varepsilon} + a_2\varepsilon & b_0 + b_1\sqrt{\varepsilon} + b_2\varepsilon \\ c_0 + c_1\sqrt{\varepsilon} + c_2\varepsilon & d_0 + d_1\sqrt{\varepsilon} + d_2\varepsilon \end{pmatrix} \in \mathbf{G}_{L,3},$$

then $g^{-1}\varphi(g) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, where

$$\begin{aligned}
a_{11} &= d_0a_0^q - b_0c_0^q + (-b_0c_1^q - b_1c_0^q + d_0a_1^q + d_1a_0^q)\sqrt{\varepsilon} + \\
&\quad (-b_0c_2^q - b_1c_1^q - b_2c_0^q + d_0a_2^q + d_1a_1^q + d_2a_0^q)\varepsilon, \\
a_{12} &= d_0b_0^q - b_0d_0^q + (-b_0d_1^q - b_1d_0^q + d_0b_1^q + d_1b_0^q)\sqrt{\varepsilon} + \\
&\quad (-b_0d_2^q - b_1d_1^q - b_2d_0^q + d_0b_2^q + d_1b_1^q + d_2b_0^q)\varepsilon, \\
a_{21} &= -c_0a_0^q + a_0c_0^q + (a_0c_1^q + a_1c_0^q - c_0a_1^q - c_1a_0^q)\sqrt{\varepsilon} + \\
&\quad (a_0c_2^q + a_1c_1^q + a_2c_0^q - c_0a_2^q - c_1a_1^q - c_2a_0^q)\varepsilon, \\
a_{22} &= -c_0b_0^q + a_0d_0^q + (a_0d_1^q + a_1d_0^q - c_0b_1^q - c_1b_0^q)\sqrt{\varepsilon} + \\
&\quad (a_0d_2^q + a_1d_1^q + a_2d_0^q - c_0b_2^q - c_1b_1^q - c_2b_0^q)\varepsilon.
\end{aligned}$$

Similarly, $g^{-1}\sigma(g) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, where

$$\begin{aligned}
b_{11} &= 1 + (b_0c_1 - b_1c_0 - d_0a_1 + d_1a_0)\sqrt{\varepsilon} + \\
&\quad (-b_0c_2 + b_1c_1 - b_2c_0 + d_0a_2 - d_1a_1 + d_2a_0)\varepsilon, \\
b_{12} &= 2(d_1b_0 - d_0b_1)\sqrt{\varepsilon}, \\
b_{21} &= 2(c_0a_1 - c_1a_0)\sqrt{\varepsilon}, \\
b_{22} &= 1 + (-d_1a_0 + d_0a_1 + b_1c_0 - b_0c_1)\sqrt{\varepsilon} + \\
&\quad (-b_0c_2 + b_1c_1 - b_2c_0 + d_0a_2 - d_1a_1 + d_2a_0)\varepsilon.
\end{aligned}$$

Hence, the condition for $g \in X$ becomes

$$\begin{cases} d_0a_0^q - b_0c_0^q + (-b_0c_1^q - b_1c_0^q + d_0a_1^q + d_1a_0^q)\sqrt{\varepsilon} + \\ (-b_0c_2^q - b_1c_1^q - b_2c_0^q + d_0a_2^q + d_1a_1^q + d_2a_0^q)\varepsilon = 1, \\ -c_0a_0^q + a_0c_0^q + (a_0c_1^q + a_1c_0^q - c_0a_1^q - c_1a_0^q)\sqrt{\varepsilon} + \\ (a_0c_2^q + a_1c_1^q + a_2c_0^q - c_0a_2^q - c_1a_1^q - c_2a_0^q)\varepsilon = \varepsilon, \\ 1 + (b_0c_1 - b_1c_0 - d_0a_1 + d_1a_0)\sqrt{\varepsilon} + \\ (-b_0c_2 + b_1c_1 - b_2c_0 + d_0a_2 - d_1a_1 + d_2a_0)\varepsilon = 1, \\ 2(c_0a_1 - c_1a_0)\sqrt{\varepsilon} = \sqrt{\varepsilon}, \\ \det(g) = 1. \end{cases}$$

Note that we have omitted two redundant equations. Now, the above system is equivalent to

$$\begin{cases} d_0a_0^q - b_0c_0^q = 1,\ a_0c_0^q = c_0a_0^q,\ a_0d_0 - b_0c_0 = 1, & (1) \\ d_0a_1^q + d_1a_0^q = b_0c_1^q + b_1c_0^q,\ d_0a_2^q + d_1a_1^q + d_2a_0^q = b_0c_2^q + b_1c_1^q + b_2c_0^q, & (2) \\ a_0c_1^q + a_1c_0^q = c_0a_1^q + c_1a_0^q,\ a_0c_2^q + a_1c_1^q + a_2c_0^q - c_0a_2^q - c_1a_1^q - c_2a_0^q = 1, & \\ b_0c_1 + d_1a_0 = b_1c_0 + d_0a_1,\ d_0a_2 - d_1a_1 + d_2a_0 = b_0c_2 - b_1c_1 + b_2c_0, & (4) \\ 2(c_0a_1 - c_1a_0) = 1, & \\ a_1d_0 + a_0d_1 = b_0c_1 + b_1c_0,\ a_0d_2 + a_1d_1 + a_2d_0 = b_0c_2 + b_1c_1 + b_2c_0. & (6) \end{cases}$$

From (1), it is easy to deduce that $a_0^q = a_0,\ c_0^q = c_0$. Using this, and subtracting the first equation (6) from the first equations in (2) and (4), and respectively, the second equation (6), from the second equations in (2) and (4), yields the equivalent system

$$\begin{cases} a_0^q = a_0,\ c_0^q = c_0,\ a_0d_0 - b_0c_0 = 1, & \\ d_0(a_1^q - a_1) = b_0(c_1^q - c_1),\ d_0(a_2^q - a_2) + d_1a_1^q - a_1d_1 = b_0(c_2^q - c_2) + b_1c_1^q - b_1c_1, & (2) \\ a_0(c_1^q - c_1) = c_0(a_1^q - a_1),\ a_0(c_2^q - c_2) - c_0(a_2^q - a_2) + a_1c_1^q - c_1a_1^q = 1, & (3) \\ a_0d_1 = b_1c_0,\ a_1d_1 = b_1c_1, & \\ 2(c_0a_1 - c_1a_0) = 1, & \\ a_1d_0 + a_0d_1 = b_0c_1 + b_1c_0,\ a_0d_2 + a_1d_1 + a_2d_0 = b_0c_2 + b_1c_1 + b_2c_0. & \end{cases}$$

Now, the first two equations in (2) and (3), together with $a_0d_0 - b_0c_0 = 1$, imply that $a_1^q = a_1,\ c_1^q = c_1$. This simplifies the other equations, so that we get

$$\begin{cases} a_0^q = a_0,\ c_0^q = c_0,\ a_0d_0 - b_0c_0 = 1, \\ a_1^q = a_1,\ c_1^q = c_1, \\ d_0(a_2^q - a_2) = b_0(c_2^q - c_2),\ a_0(c_2^q - c_2) - c_0(a_2^q - a_2) = 1, \\ a_0d_1 = b_1c_0,\ a_1d_1 = b_1c_1, \\ 2(c_0a_1 - c_1a_0) = 1, \\ a_1d_0 + a_0d_1 = b_0c_1 + b_1c_0,\ a_0d_2 + a_1d_1 + a_2d_0 = b_0c_2 + b_1c_1 + b_2c_0. \end{cases}$$

Similarly, rewriting the equations in the third row, we get the equivalent system

$$\begin{cases} a_0^q = a_0,\ c_0^q = c_0,\ a_0d_0 - b_0c_0 = 1, & \\ a_1^q = a_1,\ c_1^q = c_1, & \\ a_2^q - a_2 = b_0,\ c_2^q - c_2 = d_0, & \\ a_0d_1 = b_1c_0,\ a_1d_1 = b_1c_1, & (4) \\ 2(c_0a_1 - c_1a_0) = 1, & (5) \\ a_1d_0 + a_0d_1 = b_0c_1 + b_1c_0,\ a_0d_2 + a_1d_1 + a_2d_0 = b_0c_2 + b_1c_1 + b_2c_0. & \end{cases}$$

Now, the equations in (4) yield $a_0d_1 = b_1c_0 \Rightarrow a_0d_1a_1 = a_1b_1c_0 \Rightarrow a_0b_1c_1 = a_1b_1c_0 \Rightarrow b_1(a_0c_1 - a_1c_0) = 0$, and so by (5), we have $b_1 = 0$. Similarly, $d_1 = 0$.

Hence, our system of equations is equivalent to

$$\begin{cases} a_0^q = a_0,\ c_0^q = c_0,\ a_0d_0 - b_0c_0 = 1, \\ a_1^q = a_1,\ c_1^q = c_1,\ b_1 = 0,\ d_1 = 0, \\ a_2^q - a_2 = b_0,\ c_2^q - c_2 = d_0, \\ 2(c_0a_1 - c_1a_0) = 1, \\ a_1d_0 = b_0c_1,\ a_0d_2 + a_2d_0 = b_0c_2 + b_2c_0. \end{cases}$$

Now consider the action of the group $\mathbf{U}_2^1$. If $u = \begin{pmatrix} 1 & x\varepsilon \\ 0 & 1 \end{pmatrix} \in \mathbf{U}_2^1$, then

$$gu = \begin{pmatrix} a_0 + a_1\sqrt{\varepsilon} + a_2\varepsilon & b_0 + b_1\sqrt{\varepsilon} + (b_2 + a_0x)\varepsilon \\ c_0 + c_1\sqrt{\varepsilon} + c_2\varepsilon & d_0 + d_1\sqrt{\varepsilon} + (d_2 + c_0x)\varepsilon \end{pmatrix}.$$

Thus the set of orbits $X_L/\mathbf{U}_2^1$ can be identified with the set of points $(a_0, b_0, c_0, d_0, a_1, c_1, a_2, c_2) \in \mathbb{F}^8$, such that

$$\begin{cases} a_0^q = a_0,\ c_0^q = c_0,\ a_0d_0 - b_0c_0 = 1, \\ a_1^q = a_1,\ c_1^q = c_1, \\ a_2^q - a_2 = b_0,\ c_2^q - c_2 = d_0, \\ 2(c_0a_1 - c_1a_0) = 1, \\ a_1d_0 = b_0c_1. \end{cases} \tag{3}$$

Using the equations in (3), we eliminate b_0 and d_0, and we can identify the variety with the set of points $(a_0, c_0, a_1, c_1, a_2, c_2) \in \mathbb{F}^6$ such that

$$\begin{cases} a_0^q = a_0,\ c_0^q = c_0, \\ a_1^q = a_1,\ c_1^q = c_1, \\ 2(c_0a_1 - c_1a_0) = 1, \\ a_0(c_2^q - c_2) - c_0(a_2^q - a_2) = 1,\ a_1(c_2^q - c_2) = c_1(a_2^q - a_2). \end{cases} \tag{4}$$

The equations (4) can be rewritten so as to give the following system

$$\begin{cases} a_0^q = a_0,\ c_0^q = c_0, \\ a_1^q = a_1,\ c_1^q = c_1, \\ 2(c_0a_1 - c_1a_0) = 1, \\ f_1 = a_0c_2 - c_0a_2,\ f_2 = a_1c_2 - c_1a_2, \\ f_1^q - f_1 = 1,\ f_2^q = f_2. \end{cases}$$

In what follows, we will denote by Y_L the affine variety defined by the above system of equations. To recap, we have shown the following

Proposition 1. *The quotient $X_L/\mathbf{U}_2^1$ is canonically isomorphic to the affine variety Y_L.*

Since $\mathbf{U}_2^1$ is itself isomorphic to the affine space $\mathbb{A}^1$, the irreducible representations realised in the cohomology of Y_L are the same as those realised by X_L (cf. [11], Lemma 3.1).

Following [11], we use $(\mathbf{S}, \mathbf{S})$ to denote the subgroup of $\mathbf{G}_2$ consisting of matrices of the form

$$\begin{pmatrix} 1 + x\varepsilon & y\varepsilon \\ 0 & 1 - x\varepsilon \end{pmatrix}.$$

Recall that the group $A = \left\{ \begin{pmatrix} \pm 1 + a\varepsilon & 0 \\ 0 & \pm 1 - a\varepsilon \end{pmatrix} \mid a \in \mathbb{F}_q \right\}$ acts on X_L by right translations.

We will now show, using the results of [11], that the extended Deligne-Lusztig variety X_L realises all irreducible representations of G_2 of dimension $(q^2-1)/2$. More precisely, we show

Theorem 1. *Let $Y = \{g \in \mathbf{G}_2 \mid g^{-1}F(g) \in (\mathbf{S}, \mathbf{S})\}$. Then there is an isomorphism*

$$\alpha : Y/(\mathbf{S}, \mathbf{S}) \xrightarrow{\sim} Y_L/A,$$

which commutes with the action of G_2 on both varieties.

Proof. The condition for an element $y = \left(\begin{smallmatrix} x_0+x_1\varepsilon & y_0+y_1\varepsilon \\ z_0+z_1\varepsilon & w_0+w_1\varepsilon \end{smallmatrix}\right) \in \mathbf{G}_2$ to lie in Y is given by the equations

$$\begin{cases} x_0^q = x_0,\ y_0^q = y_0,\ z_0^q = z_0,\ w_0^q = w_0,\ x_0w_0 - y_0z_0 = 1, \\ x_0(z_1^q - z_1) = z_0(x_1^q - x_1), \\ x_1w_0 + x_0w_1 = y_0z_1 + y_1z_0, \end{cases}$$

which can be rewritten as

$$\begin{cases} x_0^q = x_0,\ y_0^q = y_0,\ z_0^q = z_0,\ w_0^q = w_0,\ x_0w_0 - y_0z_0 = 1, \\ f = x_0z_1 - z_0x_1,\ f^q = f, \\ x_1w_0 + x_0w_1 = y_0z_1 + y_1z_0. \end{cases}$$

The action on Y by an element $s = \left(\begin{smallmatrix} 1+t\varepsilon & u\varepsilon \\ 0 & 1-t\varepsilon \end{smallmatrix}\right) \in (\mathbf{S}, \mathbf{S})$ is given by

$$ys = \begin{pmatrix} x_0 + (x_1 + x_0t)\varepsilon & y_0 + (y_1 - y_0t + x_1u)\varepsilon \\ z_0 + (z_1 + z_0t)\varepsilon & w_0 + (w_1 - w_0t + z_0u)\varepsilon \end{pmatrix}.$$

Thus, the set of orbits $Y/(\mathbf{S}, \mathbf{S})$ can be identified with the set of points $(x_0, y_0, z_0, w_0, f) \in \mathbb{F}^5$, such that

$$\begin{cases} x_0^q = x_0,\ y_0^q = y_0,\ z_0^q = z_0,\ w_0^q = w_0,\ x_0w_0 - y_0z_0 = 1, \\ f^q = f. \end{cases}$$

Now for a point $(a_0, c_0, a_1, c_1, f_1, f_2) \in Y_L$, the action of an element $\left(\begin{smallmatrix} 1+a\varepsilon & 0 \\ 0 & 1-a\varepsilon \end{smallmatrix}\right) \in A$, is given in terms of coordinates by

$$(a_0, c_0, a_1, c_1, f_1, f_2) \longmapsto (a_0, c_0, a_1, c_1, f_1, f_2 + a/2).$$

Hence, the quotient Y_L/A can be identified with the set of points $(a_0, c_0, a_1, c_1, f_1) \in \mathbb{F}^5$ such that

$$\begin{cases} a_0^q = a_0,\ c_0^q = c_0, \\ a_1^q = a_1,\ c_1^q = c_1, \\ 2(c_0a_1 - c_1a_0) = 1, \\ f_1^q - f_1 = 1. \end{cases}$$

Now fix an element $\xi \in \mathbb{F}$ such that $\xi^q - \xi = 1$. Then there is clearly an isomorphism

$$\alpha : Y/(\mathbf{S}, \mathbf{S}) \longrightarrow Y_L/A,$$

given by

$$\alpha(x_0, y_0, z_0, w_0, f) = (x_0, y_0, \frac{z_0}{2}, \frac{w_0}{2}, f + \xi).$$

Note that because of the choice of ξ, this isomorphism is not canonical. It remains to show that α commutes with the action of G_2 on the varieties. Thus, let

$(x_0, y_0, z_0, w_0, f) \in Y/(\mathbf{S}, \mathbf{S})$, and $(a_0, c_0, a_1, c_1, f_1) \in Y_L/A$. Then the action of an element $\begin{pmatrix} g_0 + g_1\varepsilon & h_0 + h_1\varepsilon \\ i_0 + i_1\varepsilon & j_0 + j_1\varepsilon \end{pmatrix} \in G_2$, is given in terms of coordinates by

$$\begin{aligned}(x_0, y_0, z_0, w_0, f) \longmapsto (&g_0x_0 + h_0z_0, g_0y_0 + h_0w_0, i_0x_0 + j_0z_0, i_0y_0 + j_0w_0,\\ &f + x_0^2(g_0i_1 - i_0g_1) + x_0z_0(g_0j_1 + h_0i_1 - i_0h_1 - j_0g_1) + z_0^2(h_0j_1 - j_0h_1)),\end{aligned}$$

and respectively

$$\begin{aligned}(a_0, c_0, a_1, c_1, f_1) \longmapsto (&g_0a_0 + h_0c_0, g_0b_0 + h_0d_0, i_0a_0 + j_0c_0, i_0b_0 + j_0d_0,\\ &f_1 + a_0^2(g_0i_1 - i_0g_1) + a_0c_0(g_0j_1 + h_0i_1 - i_0h_1 - j_0g_1) + c_0^2(h_0j_1 - j_0h_1)).\end{aligned}$$

Thus, it is clear that the action of G_2 commutes with the isomorphism α, and the theorem is proved. ∎

In [11] it was shown that all irreducible representations of G_2 of dimension $(q^2 - 1)/2$ appear in the cohomology of the variety $Y/(\mathbf{S}, \mathbf{S})$. The above theorem shows that that this variety is G_2-isomorphic to a quotient of an extended Deligne-Lusztig variety by a finite group. Thus the latter variety also realises all the above representations in its cohomology.

Of course, it would be desirable to find a more conceptual proof of Theorem 1 that would not make use of explicit equations of the varieties.

5. Towards a reformulation of Lusztig's conjecture

It is clear that in the degree of generality of Section 3, the varieties we have defined may sometimes be empty sets. At the other extreme, $L_0 = F$, $\Delta = \Gamma$, and $Y_\sigma = \{1\}$ for all $\sigma \in \Delta$ gives a variety identical to G_r itself, and thus the cohomology is just the regular representation of G_r, which is not interesting for our purposes. Thus, in order to ensure a nontrivial theory and a suitable framework for constructing representations of the groups G_r, it is necessary to specialise the construction. Motivated by the construction of Deligne and Lusztig in the case $r = 1$, the results of [11], and the example of the preceding section, we suggest the following preliminary construction.

As before, let L_0/F be a finite totally ramified Galois extension of degree e, and $L = L_0^{\mathrm{ur}}$ with $\Gamma = \mathrm{Gal}(L/F)$. Let $\mathbf{G}$ be a connected reductive group over F. According to a result in the structure theory of reductive groups over local fields (cf. [10], 4.7), there exists a Borel subgroup $\mathbf{B}$ in $\mathbf{G}$, defined over F^{ur}. Let $\mathbf{U}$ denote its unipotent radical; then $\mathbf{U}$ is also defined over F^{ur}. We identify $\mathbf{G}$ and $\mathbf{U}$ with their corresponding group schemes over $\mathcal{O}_L$, obtained by extension of scalars.

A first naive extension of Lusztig's construction would be the following. Fix an integer $r \geq 1$, and a corresponding group G_r. Take Δ to be any set of topological generators of Γ, and r' any integer such that $\mathbf{G}_{L,r'}^{\Delta} = G_r$. For every $\sigma \in \Delta$, let $Y_\sigma = x_\sigma \mathbf{U}_{L,r'}$ for some $x_\sigma \in \mathbf{G}_{L,r'}$. Denote by $X_{L,r'}(\{x_\sigma\})$ the extended Deligne-Lusztig variety defined by this data, with its corresponding action of G_r. Then one can ask whether every irreducible representation of G_r appears in the l-adic cohomology of some $X_{L,r'}(\{x_\sigma\})$.

However, the cases known so far (i.e. $r = 1$, and the case discussed in Section 4) are consistent with a much more specific construction. Namely, take $\Delta = \{\varphi\} \cup \Delta'$, where Δ' is a minimal set of generators of $\mathrm{Gal}(L/F^{\mathrm{ur}})$. Since Δ topologically generates Γ, we have $\mathbf{G}_{L,r'}^{\Delta} = \mathbf{G}_{L,r'}^{\Gamma}$ for any r'. Let r' be the smallest integer such that $\mathbf{G}_{L,r'}^{\Delta} = G_r$. By Lemma 1, this means that $r' = (r-1)e + 1$. Let x_φ be a representative of double $\mathbf{B}_r$–$\mathbf{B}_r$-cosets in $\mathbf{G}_r$, and let x_σ for each $\sigma \in \Delta'$ be a representative of double $\mathbf{B}_{L,r'}$–$\mathbf{B}_{L,r'}$-cosets in $\mathbf{G}_{L,r'}$. Given this construction, the following questions present themselves:

Does every irreducible representation of G_r appear in the l-adic cohomology of some variety $X_{L,r'}(\{x_\sigma\})$?

If so:

- To what extent is the construction dependent of the choice of $\mathbf{B}$?
- Is it possible to characterise precisely what kind of extensions L_0/F that are needed? In particular, to what extent are abelian extensions enough?
- Is it always sufficient to take $r' = (r-1)e+1$, or do there exist cases where we have to take some larger r' such that $(r-1)e < r' \leq re$?
- Is it always enough to take Δ' to be a minimal set of generators of the group $\mathrm{Gal}(L/F^{\mathrm{ur}})$? How does the resulting variety depend on the choice of such a Δ'?
- For each x_φ, can the set $\{x_\sigma \mid \sigma \in \Delta'\}$ be specified further?

The answer to the first question is affirmative for $r = 1$, by the work of Deligne and Lusztig [2]. It is also affirmative for $\mathbf{G} = \mathrm{SL}_2$, F of positive characteristic, q odd, and $r = 2$, by the results in [8], sect. 3, together with the results in [11] and Theorem 1 of this paper.

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